

# A note on comparison between Birkhoff and McShane-type integrals for multifunctions

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## Abstract

Here we present some comparison results between Birkhoff and McShane multivalued integration.

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**Key words:** Birkhoff integral, McShane integral, multifunctions, Banach spaces, Rådström embedding theorem

## 1 Introduction

Several notions of multivalued integral have been developed after the pioneering papers of Aumann and Debreu in the sixties and they are used extensively in economic theory and optimal control, we cite here for example [1, 2, 6]. In [5] Boccuto and Sambucini and in [10, 11, 8, 9] Cascales, Kadets and Rodríguez introduced the McShane and the Birkhoff multivalued integrals respectively, while in [13, 14] Di Piazza and Musiał introduced the Kurzweil-Henstock one. Since these kinds of integration lie strictly between Bochner and Pettis integrability (both in the single-valued and in multivalued cases) it is natural to study the possible relationships between the Birkhoff and McShane integrals and with the other multivalued integrals, in particular with the Pettis and Aumann Pettis studied also in [3, 4, 15]. This paper is organized as follows: in Section 2 we recall some known results for the single-valued case, in Section 3 we recall multivalued integrals, while

in Section 4 we give the comparison results between them and with respect to the Aumann integral obtained with different types of selections.

The results that we obtain are the following: the McShane and the Birkhoff multivalued integrals are equivalent in Banach spaces with weak\* separable dual unit ball (for example in separable Banach spaces) and in this case they agree also with the Aumann Pettis multivalued integrals (this last comparison for the McShane multivalued integral holds without separability assumption).

## 2 The single-valued McShane and Birkhoff integrals

Throughout this paper  $\Omega$  is an abstract non-empty set and  $\mathcal{T}$  is a topology on  $\Omega$  making  $(\Omega, \mathcal{T}, \Sigma, \mu)$  a  $\sigma$ -finite quasi-Radon measure space which is *outer regular*, namely such that

$$\mu(B) = \inf\{\mu(G) : B \subseteq G \in \mathcal{T}\} \quad \text{for all } B \in \Sigma.$$

A *generalized McShane partition*  $P$  of  $\Omega$  ([18, Definitions 1A]) is a disjoint sequence  $(E_i, t_i)_{i \in \mathbb{N}}$  of measurable sets of finite measure, with  $t_i \in \Omega$  for every  $i \in \mathbb{N}$  and  $\mu(\Omega \setminus \bigcup_i E_i) = 0$ .

A *gauge* on  $\Omega$  is a function  $\Delta : \Omega \rightarrow \mathcal{T}$  such that  $s \in \Delta(s)$  for every  $s \in \Omega$ . A generalized McShane partition  $(E_i, t_i)_i$  is  $\Delta$ -*fine* if  $E_i \subset \Delta(t_i)$  for every  $i \in \mathbb{N}$ .

From now on, let  $X$  be a Banach space, denote by the symbol  $\mathcal{P}$  the class of all generalized McShane partitions of  $\Omega$  and by  $\mathcal{P}_\Delta$  the set of all  $\Delta$ -fine elements of  $\mathcal{P}$ .

**Definition 2.1** A function  $f : \Omega \rightarrow X$  is said to be

**2.1.1** *McShane integrable*, with integral  $w$ , if for every  $\varepsilon > 0$  there exists a gauge  $\Delta : \Omega \rightarrow \mathcal{T}$  such that

$$\limsup_{n \rightarrow +\infty} \left\| w - \sum_{i=1}^n \mu(E_i) f(t_i) \right\| \leq \varepsilon$$

for every  $\Delta$ -fine McShane partition  $(E_i, t_i)_i$ . In this case, we write  $\int_\Omega f = w$  (see [18, Definition 1A]);

**2.1.2** *Birkhoff integrable*, if for every  $\varepsilon > 0$  there exists a countable partition  $\Gamma = (A_n)_n$  of  $\Omega$  in  $\Sigma$ , for which  $f$  is summable (namely  $J(f, \Gamma) :=$

$\{\sum_n \mu(A_n)f(t_n) : t_n \in A_n\}$  is made up of unconditionally convergent series) and  $\sup_{x,y \in J(f,\Gamma)} \|x - y\| \leq \varepsilon$ . In this case the Birkhoff integral of  $f$  is

$$(B) \int_{\Omega} f d\mu = \bigcap \{ \overline{co(J(f,\Gamma))} : f \text{ is summable with respect to } \Gamma \}$$

(see [7], [17, (b), Section 4]).

The Birkhoff integral was also defined for  $\sigma$ -finite measure spaces ([7]) considering only partitions into sets of finite measure. We now recall the known results on the McShane and Birkhoff integrals in the single-valued case.

**Remark 2.2** (See for reference [17, 18, 19, 21, 22, 25, 31]) Let  $\Omega$  be a quasi-Radon probability space. If  $f$  is Birkhoff integrable, then  $f$  is McShane integrable and the respective integrals coincide; if  $B_{X^*}$  is separable in the weak\*-topology (e.g. when  $X$  is separable), then a function  $f$  is Birkhoff integrable if and only if  $f$  is McShane integrable ([19, Theorem 10]). Birkhoff integrability is in general stronger than McShane integrability, since it is possible to construct a bounded McShane integrable function  $f : [0, 1] \rightarrow l_{\infty}([0, 1])$  which is not Birkhoff integrable (see [19, §8, Example]). If we compare these definitions of integrability with the other ones known in the literature we remember that Bochner integrability implies McShane integrability and the two integrals are equal ([18, Theorem 1K]), while McShane integrability implies Pettis integrability and the two integrals coincide ([18, Theorem 1Q]). Finally, if the Banach space  $X$  is separable, then Birkhoff, McShane and Pettis integrability are equivalent ([18, Corollary 4C], [27]). For a more detailed investigation on the properties of the Birkhoff and Pettis integrals see also [8, 11, 26].

### 3 Multivalued integrals

We skip now to the multivalued case. Independently in [5] and in [10] the authors studied two kinds of multivalued integration related to McShane and Birkhoff integrability using a Rådström embedding theorem and compared them with the usual Aumann integral.<sup>1</sup>

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<sup>1</sup>We point out that all the results given in [5] are expressed for  $\Omega = [a, b]$ , where  $a, b \in [-\infty, +\infty]$ ,  $a < b$ , only for the sake of simplicity. Moreover,  $\mathcal{T}$ ,  $\Sigma$  and  $\mu$  are the families of all open subsets of  $[a, b]$ , the  $\sigma$ -algebra of all Lebesgue measurable subsets of

Let  $X$  be a Banach space,  $cwk(X)$   $[ck(X)]$  denote the family of all convex and weakly compact [respectively convex and compact] subsets of  $X$ . We denote by the symbol  $d(x, C)$  the usual distance between a point and a nonempty set  $C \subset X$ , namely  $d(x, C) = \inf\{\|x - y\| : y \in C\}$ ,  $\delta^*(x^*, A) = \sup_{x \in A} x^*(x)$  and with  $h$  the usual Hausdorff distance. We recall also that a multifunction  $F$  is *measurable* if  $F^-(C)$  is a Borel set for every closed set  $C \subset X$ , and that  $F$  is *integrably bounded* if there exists  $g \in L^1(\Omega)$  such that  $h(F(t), \{0\}) \leq g(t)$   $\mu$ -a.e. .

Thanks to the Rådström embedding theorem (see [28]),  $cwk(X)$  endowed with the Hausdorff distance  $h$  is a complete metric space that can be isometrically embedded into a Banach space, for example in the Banach space of bounded real valued functions defined on  $B_{X^*}$ ,  $l_\infty(B_{X^*})$  endowed with the supremum norm  $\|\cdot\|_\infty$  by means of the mapping  $j : cwk(X) \mapsto l_\infty(B_{X^*})$  given by  $j(A) := \delta^*(\cdot, A)$  (see [12, Theorems II.18 and II.19] and [10, Lemma 1.1] for the notations). So the authors in [5] and [10] defined the multivalued integrals as follows:

**Definition 3.1** Let  $F : \Omega \rightarrow cwk(X)$  be a multifunction. For every  $A \in \Sigma$  we say that  $F$  is:

**(3.1.1)** *McShane integrable* if there exists  $I \in cwk(X)$  such that for every  $\varepsilon > 0$  there exists a gauge  $\Delta$  such that  $\limsup_n h(I, \sum_{i=1}^n F(t_i)\mu(E_i)) \leq \varepsilon$  for every generalized  $\mathcal{P}_\Delta$  McShane partition  $\Pi = (E_i, t_i)_i$  of  $A$ . In this case the McShane integral of  $F$  on  $A$  is defined by:  $I := (McS) \int_A F(t) d\mu$  ([5, Definition 1]);

**(3.1.2)** *Birkhoff integrable* if the single-valued function  $j \circ F : \Omega \rightarrow l_\infty(B_{X^*})$  is Birkhoff integrable. Since  $j(cwk(X))$  is a closed convex cone in  $l_\infty(B_{X^*})$ ,  $\int_A j \circ F d\mu \in cwk(X)$ , and therefore there is a unique element  $(B) \int_A F d\mu \in cwk(X)$ , called the Birkhoff integral of  $F$  on  $A$ , which satisfies

$$j \left( (B) \int_A F d\mu \right) = \int_A j \circ F d\mu$$

([10, Definition 2.1]);

**(3.1.3)** *Aumann integrable* if

$$(A) \int_A F d\mu = \left\{ \int_A f d\mu, f \in \mathcal{C} \right\} \neq \emptyset,$$

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$[a, b]$  and the Lebesgue measure on  $[a, b]$  respectively. We observe that all the results given there hold as well whenever  $\Omega$  is any non-empty  $\sigma$ -finite quasi Radon outer regular measure space.

where for  $\mathcal{C}$  we consider the following sets:  $S_F^1, S_{McS}^1, S_B^1, S_{Pe}^1$  (the sets of all Bochner, McShane, Birkhoff and Pettis integrable selections of  $F$  respectively);

**(3.1.4)** *Pettis integrable* if  $\delta^*(x^*, F) : \Omega \rightarrow \mathbb{R}$  given by  $\delta^*(x^*, F)(\omega) = \delta^*(x^*, F(\omega))$  is  $\mu$ -integrable and if for every  $A \in \Sigma$  there exists  $C_A \in cwk(X)$  such that

$$\delta^*(x^*, C_A) = \int_A \delta^*(x^*, F(\omega)) d\mu \text{ for every } x^* \in X^*.$$

In this case we write  $C_A = (P) \int_A F d\mu$ . See [15, Theorem 5.4] for a number of equivalent definitions.

We remark moreover that, in [10, Corollary 2.7] it is observed that the definition of the Birkhoff integral does not depend on the particular embedding used.

## 4 Comparisons between multivalued integrals

What we obtain in this paper is a comparison between the different types of multivalued integrals introduced before. First of all we report some equivalence conditions for both the McShane and the Birkhoff integral:

**(4.a)** If  $F : \Omega \rightarrow cwk(X)$  is McShane integrable, then its integral coincides with the  $(\star)$ -integral, namely  $(McS) \int_{\Omega} F d\mu = \Phi(F, \Omega)$ , where

$$\Phi(F, \Omega) = \{x \in X : \forall \varepsilon > 0, \exists \text{ a gauge } \Delta : \text{ for every generalized } \mathcal{P}_{\Delta} \text{ McShane partition } (E_i, t_i)_{i \in \mathbb{N}} \text{ there holds:}$$

$$\limsup_n d(x, \sum_{i=1}^n F(t_i) \mu(E_i)) \leq \varepsilon\}.$$

(See [5, Proposition 1]). No measurability is required a priori and so we can define the multivalued integral also in non separable Banach spaces; moreover, if  $F$  is single-valued, then  $\Phi(F, \Omega)$  coincides with the classical McShane integral, if it exists.

**(4.b)** The Birkhoff integrability of  $F$  is equivalent to:

- (i) there is  $W \in cwk(X)$  with the following property: for every  $\varepsilon > 0$  there exists a countable partition  $\Gamma_0$  of  $\Omega$  in  $\Sigma$  such that for every countable partition  $\Gamma = \{A_n\}$  of  $\Omega$  in  $\Sigma$  finer than  $\Gamma_0$  and any choice of points  $t_n$  in  $\Gamma$ ,  $n \in \mathbb{N}$ , the series  $\sum_{n=1}^{\infty} \mu(A_n)F(t_n)$  is unconditionally convergent and  $h(\sum_{n=1}^{\infty} \mu(A_n)F(t_n), W) \leq \varepsilon$ . In this case,  $W = (B) \int_{\Omega} F d\mu$ .

(see [10, Proposition 2.6]). Moreover in [10, Proposition 2.9] it is showed that for bounded multifunctions  $F$  the Birkhoff integrability is equivalent to both Birkhoff and Bourgain properties.

As said in Remark 2.2, the Birkhoff and McShane single-valued integrals coincide when the Banach space has weak\* separable dual unit ball. The following simple lemma is the key to extend this result from the single-valued case to the case of  $cwk(X)$ -valued functions.

If  $A$  is a subset of a real vector space  $V$ , we denote by  $\text{aco}_{\mathbb{Q}}(A)$  (resp.  $\text{aco}(A)$ ) the set of all elements  $v \in V$  that can be written as  $v = \sum_{i=1}^n \lambda_i v_i$ ,  $n \in \mathbb{N}$ , with  $v_i \in A$ ,  $\lambda_i \in \mathbb{Q}$  (resp.  $\lambda_i \in \mathbb{R}$ ) and  $\sum_{i=1}^n |\lambda_i| \leq 1$ .

**Lemma 4.1** *Let  $Y := C_b(B_{X^*})$  be the Banach space (with the supremum norm) of all real bounded and continuous functions on  $(B_{X^*}, \tau)$ , where  $\tau$  is the restriction of the Mackey topology in  $X^*$ . If  $B_{X^*}$  is weak\* separable, then  $B_{Y^*}$  is weak\* separable.*

**Proof:** Since  $B_{X^*}$  is weak\* separable, there is a countable set  $D \subset B_{X^*}$  such that  $\overline{D}^{\tau} = B_{X^*}$  (see the proof of [10, Lemma 3.6]). Given  $d \in D$ , let us consider the element  $y_d^* \in B_{Y^*}$  defined by  $y_d^*(f) := f(d)$ . Since  $\{y_d^* : d \in D\} \subset B_{Y^*}$  is norming, by applying the Hahn-Banach theorem we get

$$\overline{\text{aco}(\{y_d^* : d \in D\})}^{\text{weak}^*} = B_{Y^*},$$

and hence

$$\overline{\text{aco}_{\mathbb{Q}}(\{y_d^* : d \in D\})}^{\text{weak}^*} = B_{Y^*}.$$

Therefore  $\text{aco}_{\mathbb{Q}}(\{y_d^* : d \in D\})$  is a countable weak\* dense subset of  $B_{Y^*}$ , and the proof is complete.  $\square$

**Corollary 4.2** *Let  $\Omega$  be a quasi-Radon probability space and let  $X$  be a Banach space such that  $B_{X^*}$  is weak\* separable. Then a multi-valued function  $F : \Omega \rightarrow cwk(X)$  is McShane integrable if and only if  $F$  is Birkhoff integrable. In this case, the two integrals coincide.*

**Proof:** Let  $j : cwk(X) \rightarrow l_\infty(B_{X^*})$  be the embedding used in [10]. Then it is easy to see that  $F : \Omega \rightarrow cwk(X)$  is McShane integrable if and only if the single-valued function  $j \circ F : \Omega \rightarrow l_\infty(B_{X^*})$  is McShane integrable according to [18, Definitions 1A], and in this case  $j(\int F) = \int j \circ F$ . Since  $j(cwk(X)) \subset C_b(B_{X^*})$ , the function  $j \circ F$  takes values in  $C_b(B_{X^*})$ , which is a closed subspace of  $l_\infty(B_{X^*})$  with weak\* separable dual unit ball by virtue of Lemma 4.1. By [19, Theorem 10]  $j \circ F$  is McShane integrable if and only if  $j \circ F$  is Birkhoff integrable and the respective integrals coincide.  $\square$

The same conclusion of Corollary 4.2 can be obtained for  $\sigma$ -finite quasi-Radon outer regular measure spaces.

We now want to compare the McShane and the Birkhoff multivalued integrals with the Aumann integral, when the multifunction  $F$  has some kind of measurability. For the case of Birkhoff integrability the result is given in [10, Proposition 3.1], for the McShane case we have:

**Theorem 4.3** *Let  $F : \Omega \rightarrow cwk(X)$  be a McShane integrable multifunction, then  $F$  is Pettis integrable and for every  $A \in \Sigma$  we have*

$$(McS) \int_A F d\mu = \overline{\left\{ \int_A f d\mu, f \in S_{Pe}^1 \right\}}. \quad (1)$$

Moreover, if  $(\Omega, \mathcal{T}, \Sigma, \mu)$  is a Radon measure space or there is no real-valued-measurable cardinal and every Pettis integrable selection  $f$  is measurable (that is,  $f^{-1}(G) \in \Sigma$  for every norm-open set  $G \subseteq X$ ), then

$$(McS) \int_A F d\mu = \overline{\left\{ \int_A f d\mu, f \in S_{McS}^1 \right\}}. \quad (2)$$

If  $X$  is separable then

$$\begin{aligned} (B) \int_A F d\mu &= \left\{ \int_A f d\mu, f \in S_B^1 \right\} = \left\{ \int_A f d\mu, f \in S_{McS}^1 \right\} \\ &= (McS) \int_A F d\mu. \end{aligned} \quad (3)$$

Finally, if  $F$  is measurable and integrably bounded and  $X$  is separable and there exists a countable family  $(x_n^*)_n$  in  $X^*$  which separates points of  $X$ , then

$$(McS) \int_A F d\mu = \left\{ \int_A f d\mu, f \in S_F^1 \right\}. \quad (4)$$

**Proof:** If  $F$  is McShane integrable then it means that  $j \circ F$  is McShane integrable thanks to the Rådström embedding theorem. So  $j \circ F$  is Pettis integrable, and then, by [9, Proposition 4.4],  $F$  is Pettis integrable and for all  $A \in \Sigma$  and  $x^* \in B_{X^*}$  we get:

$$\delta^*(x^*, F) = \langle e_{x^*}, j \circ F \rangle \in L^1(\mu);$$

where  $e_{x^*} \in B_{l_\infty(B_{X^*})^*}$  is defined by:  $\langle e_{x^*}, g \rangle := g(x^*)$  for every  $g \in l_\infty(B_{X^*})$ . Then

$$\begin{aligned} \delta^* \left( x^*, (McS) \int_A F d\mu \right) &= \langle e_{x^*}, j \left( (McS) \int_A F d\mu \right) \rangle = \langle e_{x^*}, \int_A j \circ F d\mu \rangle \\ &= \int_A \langle e_{x^*}, j \circ F \rangle d\mu = \int_A \delta^*(x^*, F) d\mu. \end{aligned}$$

Thus for all  $A \in \Sigma$  we have

$$(P) \int_A F d\mu = (McS) \int_A F d\mu.$$

Now by [9, Theorems 2.5 and 2.6]  $F$  admits Pettis integrable selections and

$$(P) \int_A F d\mu = \overline{\left\{ \int_A f d\mu, f \in S_{P_e}^1 \right\}},$$

which proves (1). Now, by virtue of [18, Theorem 1Q and Corollary 4D], in our context McShane and Pettis integrability coincide for single-valued functions and this proves (2).

Observe that the concept of no real-valued measurable cardinal which appears in [18, Corollary 4D] and in [16, (e)] is contained in [20, §438] using the new terminology measure-free cardinals. In this case all metric spaces are Radon and  $f$  is McShane integrable using [20, 438D] and [18, Corollary 2G].

If  $X$  is separable, the first equality in (3) for the Birkhoff integral is given in [10, Proposition 3.1] and the last equalities follow in an analogous way and taking into account the equivalence among Pettis, Birkhoff and McShane integrability. So we obtain again the equivalence between the Birkhoff and the McShane multivalued integrals in a different way. Finally (4) is given in [5, Theorem 1].  $\square$

In the end the comparison with the Debreu integral is obvious thanks to the given definitions (see [10, Proposition 3.1 (i), Theorem 3.2] and [5, page



321 and Corollary 1]). Comparisons between Aumann and Debreu integrals are given also in [23, 24, 29, 30].

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